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COMMENT

Normally ordering some multimode exponential operators by virtue of the IWOP technique

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Abstract. This work shows that the technique of integration within an ordered product (IWOP) provides us with a very simple approach to deriving the normal product form of some multimode exponential operators, which greatly simplifies the calculations of normalising some state vectors in Hilbert space.

1. Introduction

As is well known, many normally ordered exponential operators [1] are widely used in quantum mechanics. Recently, a convenient approach to normally reordering some exponential operators is developed and is called the technique of integration within an ordered product (IWOP) [2-4]. For example, let a_i be the annihilation operator of a harmonic oscillator, satisfying the commutator $[a_i, a_j^\dagger] = \delta_{ij}$ $(i, j = 1, 2, \ldots, n)$, with a_i annihilating the vacuum state $|0\rangle_i$, whose normalised eigenstate (coherent state) [5] is denoted as $|z_i\rangle = \exp[z_i a_i^\dagger - z_i^* a_i]|0\rangle_i$, which possesses the overcomplete relation

$$\int \frac{d^2 z_i}{\pi} |z\rangle_{ii}\langle z| = \int \frac{d^2 z_i}{\pi} :\exp[-|z_i|^2 + z_i a_i^\dagger + z_i^* a_i - a_i^\dagger a_i]: = 1 \qquad a_i|z_i\rangle = z_i|z_i\rangle \tag{1}$$

where : : stands for the normal product and we used

$$|0\rangle_{ii}\langle 0| = :e^{-a_i^\dagger a_i}: \tag{2}$$

as well as the following integration formula

$$\int \frac{d^2 z}{\pi} \exp[-\zeta|z|^2 + \xi z + \eta z^*] = \zeta^{-1} \exp\left[\frac{\xi\eta}{\zeta}\right] \qquad \operatorname{Re}\zeta > 0. \tag{3}$$

Using (3) and the IWOP we can readily put the operator $e^{fa_1a_2} e^{ga_1^\dagger a_2^\dagger}$ into the normal product form [6]

$$\begin{aligned}
e^{fa_1a_2} e^{ga_1^\dagger a_2^\dagger} &= \int \frac{d^2 z_1\, d^2 z_2}{\pi^2} e^{fa_1a_2}|z_1 z_2\rangle\langle z_1 z_2| e^{ga_1^\dagger a_2^\dagger} \\
&= \int \frac{d^2 z_1\, d^2 z_2}{\pi^2} :\exp[-|z_1|^2 - |z_2|^2 + z_1 a_1^\dagger + z_2 a_2^\dagger - a_2^\dagger a_2 \\
&\quad + z_1^* a_1 - a_1^\dagger a_1 + z_2^* a_2 + f z_1 z_2 + g z_1^* z_2^*]: \\
&= \frac{1}{1-fg} \exp\left[\frac{g a_1^\dagger a_2^\dagger}{1-fg}\right] :\exp\left[\frac{fg}{1-fg}(a_1^\dagger a_1 + a_2^\dagger a_2)\right]: \exp\left[\frac{f a_1 a_2}{1-fg}\right]
\end{aligned} \tag{4}$$

where $\operatorname{Re}(1-fg) > 0$ guarantees a convergent integration in (4).

The method of derivation of (4) is, of course, quite simple and thus motivates us to tackle some other more complicated exponential operators. In sections 2 and 3 we try to put some multimode exponential operators into normally ordered form by exploiting the IWOP technique and the completeness relations of coordinate, momentum and the coherent state representations. In so doing, we can obtain some new operator identities. In section 4 we show that these new identities can facilitate the calculations of the normalisation factors of some state vectors in Hilbert space.

2. The normally ordered form of $\exp[a_i\sigma_{ij}a_j]\exp[a_i^\dagger\tau_{ij}a_j^\dagger]$

In the following we shall adopt the Einstein convention—if an index is repeated in a term, summation over it from 1 to n is implied.

Consider then how to normally reorder the operator $S\equiv\exp[a_i\sigma_{ij}a_j]\exp[a_i^\dagger\tau_{ij}a_j^\dagger]$, where σ and τ are both $n\times n$ symmetric matrices. No research has so far tackled this problem. In this section we show that the IWOP and the following integral formula in a complex Hilbert space can solve this problem. The formula is

$$I\equiv\int\prod_i^n\left[\frac{d^2z_i}{\pi}\right]\exp\left[-\tfrac{1}{2}(z\quad z^*)\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}z\\ z^*\end{pmatrix}+(\mu\quad\nu^*)\begin{pmatrix}z\\ z^*\end{pmatrix}\right]$$
$$=\left[\det\begin{pmatrix}C & D\\ A & B\end{pmatrix}\right]^{-1/2}\exp\left[\tfrac{1}{2}(\mu\quad\nu^*)\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \nu^*\end{pmatrix}\right]\tag{5}$$

where A, B, C, D are all square matrices of order n, $B=\tilde{B}$, $C=\tilde{C}$ and

$$(z\quad z^*)\equiv(z_1\ z_2\ldots z_n\ z_1^*\ z_2^*\ldots z_n^*).\tag{6}$$

Equation (5) can be found in [7]. The existing condition for this integral is also discussed in some detail by Berezin [7]. In all the following integration calculations, wherever necessary, we assume the conditions are satisfied for stressing the main point of our technique. An equivalent form of (5) is

$$I=\left[\det\begin{pmatrix}C & D\\ A & B\end{pmatrix}\right]^{-1/2}\exp\left[\tfrac{1}{2}(\mu\quad\nu^*)\begin{pmatrix}C & D\\ A & B\end{pmatrix}^{-1}\begin{pmatrix}\nu^*\\ \mu\end{pmatrix}\right]\tag{7}$$

where we used

$$\begin{pmatrix}\nu^*\\ \mu\end{pmatrix}=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\mu\\ \nu^*\end{pmatrix}\qquad\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}^{-1}=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\qquad \mathbb{1}: n\times n \text{ unit matrix.}\tag{8}$$

As a result of (7) and the IWOP we are able to put S into the normal product form, e.g.

$$S=\int\prod_{i=1}^n\left(\frac{d^2z_i}{\pi}\right)\exp[a_i\sigma_{ij}a_j]|z_1\ldots z_n\rangle\langle z_1\ldots z_n|\exp[a_i^\dagger\tau_{ij}a_j^\dagger]$$
$$=\int\prod_i^n\left(\frac{d^2z_i}{\pi}\right):\exp[-z_i^*z_i+a_i^\dagger z_i+a_iz_i^*+z_i\sigma_{ij}z_j+z_i^*\tau_{ij}z_j^*-a_i^\dagger a_i]:$$
$$=\int\prod_i^n\left(\frac{d^2z_i}{\pi}\right):\exp\left[-\tfrac{1}{2}(z\quad z^*)\begin{pmatrix}-2\sigma & \mathbb{1}\\ \mathbb{1} & -2\tau\end{pmatrix}\begin{pmatrix}z\\ z^*\end{pmatrix}+(a^\dagger\quad a)\begin{pmatrix}z\\ z^*\end{pmatrix}-a_i^\dagger a_i\right]:$$
$$=\left[\det\begin{pmatrix}\mathbb{1} & -2\tau\\ -2\sigma & \mathbb{1}\end{pmatrix}\right]^{-1/2}:\exp\left[\tfrac{1}{2}(a^\dagger\quad a)\begin{pmatrix}\mathbb{1} & -2\tau\\ -2\sigma & \mathbb{1}\end{pmatrix}^{-1}\begin{pmatrix}a\\ a^\dagger\end{pmatrix}-a_i^\dagger a_i\right]:\tag{9}$$

where $(a^\dagger \quad a)=(a_1^\dagger\, a_2^\dagger \ldots a_n^\dagger\, a_1\, a_2 \ldots a_n)$, σ and τ should be such as to give a convergent integral as demanded by [7]. According to the method of partitioning of matrices for finding the inverse and the determinant of a matrix [8]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}=\begin{pmatrix} (A-BD^{-1}C)^{-1} & A^{-1}B(CA^{-1}B-D)^{-1} \\ D^{-1}C(BD^{-1}C-A)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix} \tag{10}$$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\det A \det(D-CA^{-1}B) \tag{11}$$

where, wherever necessary, we assume the matrices are non-singular, we have

$$\begin{aligned}\begin{pmatrix} \mathbb{1} & -2\tau \\ -2\sigma & \mathbb{1} \end{pmatrix}^{-1}&=\begin{pmatrix} (\mathbb{1}-4\tau\sigma)^{-1} & -2\tau(4\sigma\tau-\mathbb{1})^{-1} \\ -2\sigma(4\tau\sigma-\mathbb{1})^{-1} & (\mathbb{1}-4\sigma\tau)^{-1} \end{pmatrix} \\ &=\begin{pmatrix} (\mathbb{1}-4\tau\sigma)^{-1} & (\mathbb{1}-4\tau\sigma)^{-1}2\tau \\ (\mathbb{1}-4\sigma\tau)^{-1}2\sigma & (\mathbb{1}-4\sigma\tau)^{-1} \end{pmatrix}.\end{aligned} \tag{12}$$

Thus (9) can be further expressed as

$$\begin{aligned}S=[\det(\mathbb{1}-4\sigma\tau)]^{-1/2}\exp\{a_i^\dagger[(\mathbb{1}-4\tau\sigma)^{-1}\tau]_{ij}a_j^\dagger\} :\exp[a_i^\dagger(\mathbb{1}-4\tau\sigma)_{ij}^{-1}a_j-a_i^\dagger a_i]: \\ \times\exp\{a_i[(\mathbb{1}-4\sigma\tau)^{-1}\sigma]_{ij}a_j\}.\end{aligned} \tag{13}$$

Using the operator identity [9]

$$\exp(a_i^\dagger\Lambda_{ij}a_j)=:\exp[a_i^\dagger(e^\Lambda-\mathbb{1})_{ij}a_j]: \tag{14}$$

we can rewrite the second exponential operator in (13) as

$$:\exp\{a_i^\dagger[(\mathbb{1}-4\tau\sigma)^{-1}-\mathbb{1}]_{ij}a_j\}:=\exp\{a_i^\dagger[\ln(\mathbb{1}-4\tau\sigma)^{-1}]_{ij}a_j\}. \tag{15}$$

In particular, if $2\sigma_{12}=f$, $2\tau_{12}=g$, then (13) reduces to (4). As another example of using (13), we examine the following case:

$$\sigma=\frac{1}{2}\begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & \rho \\ 0 & \rho & 0 \end{pmatrix} \qquad \tau=\frac{1}{2}\begin{pmatrix} 0 & \lambda' & 0 \\ \lambda' & 0 & \rho' \\ 0 & \rho' & 0 \end{pmatrix}. \tag{16}$$

It is easy to calculate

$$\begin{aligned}(\mathbb{1}-4\tau\sigma)^{-1}\tau=\frac{\tau}{(1-\lambda\lambda'-\rho\rho')} \qquad (\mathbb{1}-4\sigma\tau)^{-1}\sigma=\frac{\sigma}{(1-\lambda\lambda'-\rho\rho')} \\ (\mathbb{1}-4\tau\sigma)^{-1}=\begin{pmatrix} 1-\rho\rho' & 0 & \lambda'\rho \\ 0 & 1 & 0 \\ \rho'\lambda & 0 & 1-\lambda'\lambda \end{pmatrix}(1-\lambda\lambda'-\rho\rho')^{-1}.\end{aligned} \tag{17}$$

Therefore, from (13) we directly obtain

$$\begin{aligned}&e^{\lambda a_1 a_2+\rho a_2 a_3}\, e^{\lambda' a_1^\dagger a_2^\dagger+\rho' a_2^\dagger a_3^\dagger} \\ &\quad=\exp\left[(a_1 \quad a_2 \quad a_3)\sigma\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right]\exp\left[(a_1^\dagger \quad a_2^\dagger \quad a_3^\dagger)\tau\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \end{pmatrix}\right]\end{aligned}$$

$$= \frac{1}{1-\lambda\lambda'-\rho\rho'} \exp\left(\frac{\lambda' a_1^\dagger a_2^\dagger + \rho' a_2^\dagger a_3^\dagger}{1-\lambda\lambda'-\rho\rho'}\right)$$

$$\times :\exp\left\{\frac{1}{1-\lambda\lambda'-\rho\rho'}[a_1^\dagger a_1 \lambda\lambda' + a_2^\dagger a_2(\lambda\lambda'+\rho\rho')\right.$$

$$\left. + a_3^\dagger a_3 \rho\rho' + \lambda'\rho a_1^\dagger a_3 + \rho'\lambda a_3^\dagger a_1]\right\}: \exp\left(\frac{\lambda a_1 a_2 + \sigma a_2 a_3}{1-\lambda\lambda'-\rho\rho'}\right). \tag{18}$$

In the third example let us consider what is the normal product form of $W \equiv \exp[K a_1^2 + \delta a_1 a_2] \exp[K' a_1^{\dagger 2} + \delta' a_1^\dagger a_2^\dagger]$; in this case

$$\sigma = \begin{pmatrix} K & \delta/2 \\ \delta/2 & 0 \end{pmatrix} \qquad \tau = \begin{pmatrix} K' & \delta'/2 \\ \delta'/2 & 0 \end{pmatrix}. \tag{19}$$

It then follows that

$$(1-4\sigma\tau)^{-1}\sigma = \begin{pmatrix} K & \delta(1-\delta\delta')/2 \\ \delta(1-\delta\delta')/2 & K'\delta^2 \end{pmatrix} Y^{-1}$$

$$(1-4\tau\sigma)^{-1}\tau = \begin{pmatrix} K' & \delta'(1-\delta\delta')/2 \\ \delta'(1-\delta\delta')/2 & K\delta'^2 \end{pmatrix} Y^{-1} \qquad Y \equiv (1-\delta\delta')^2 - 4KK'. \tag{20}$$

Therefore

$$W = Y^{-1/2} \exp\{Y^{-1}[K' a_1^{\dagger 2} + \delta'^2 K a_2^{\dagger 2} + \delta'(1-\delta\delta') a_1^\dagger a_2^\dagger]\}$$

$$\times :\exp\left\{(a_1^\dagger \quad a_2^\dagger) Y^{-1} \begin{pmatrix} 1-\delta\delta' & 2K'\delta \\ 2K\delta' & 1-4KK'-\delta\delta' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - a_1^\dagger a_1 - a_2^\dagger a_2 \right\}:$$

$$\times \exp\{Y^{-1}[K a_1^2 + \delta^2 K' a_2^2 + \delta(1-\delta\delta') a_1 a_2]\}. \tag{21}$$

In particular, when $\delta = \delta' = 0$, (21) reduces to

$$W|_{\delta=\delta'=0} = \frac{1}{\sqrt{1-4KK'}} \exp\left(\frac{K' a_1^{\dagger 2}}{1-4KK'}\right) :\exp\left\{\left(\frac{1}{1-4KK'} - 1\right) a_1^\dagger a_1\right\}:$$

$$\times \exp\left(\frac{K a_1^2}{1-4KK'}\right). \tag{22}$$

The operator identities (13), (18) and (21) seem to be first reported here. It must be emphasised that in deriving them we have used the existing condition of the integration (9); for example, (18) is derived under the condition: $\mathrm{Re}(\lambda\lambda') < 1$ and $\mathrm{Re}(\rho\rho'/(1-\lambda\lambda')) < 1$, whereas (22) is deduced under the condition: $\mathrm{Re}(-1+K+K') < 0$, $\mathrm{Re}[(1-4KK')/(-1+K+K')] < 0$; or $\mathrm{Re}(-1-K-K') < 0$, $\mathrm{Re}[(1-4KK')/(-1-K-K')] < 0$.

3. Normally ordered expansion of $\exp[\hat{x}_l \Lambda_{lm} \hat{x}_m]$ and $\exp[\hat{P}_l \Lambda_{lm} \hat{P}_m]$

In this section we aim at transforming $\exp[\hat{x}_l \Lambda_{lm} \hat{x}_m]$ and $\exp[\hat{P}_l \Omega_{lm} \hat{P}_m]$ into normal product by exploiting the completeness relations of coordinate and momentum representations, where Λ and Ω are both 3×3 symmetric matrices. Let $|x\rangle$ ($|p\rangle$) be a coordinate (momentum) eigenstate with $x=(x_1,x_2,x_3)$ ($p=(p_1,p_2,p_3)$) in three-dimensional Euclidean space. As is well known, the representation theory of Dirac [10] tells us

$$\int \mathrm{d}x|x\rangle\langle x|=1 \qquad \hat{x}_l|x\rangle=x_l|x\rangle \qquad \mathrm{d}x\equiv \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \qquad \hat{x}_l=\sqrt{\frac{\hbar}{2M\omega}}(a_l+a_l^\dagger)$$

$$\int \mathrm{d}p|p\rangle\langle p|=1 \qquad \hat{P}_l|p\rangle=p_l|p\rangle \qquad \mathrm{d}p\equiv \mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3 \qquad \hat{P}_l=\sqrt{\frac{M\omega\hbar}{2}}(a_l-a_l^\dagger)/i.$$

$$l=(1,2,3) \tag{24}$$

Using the Fock representation of $|x\rangle$ and $|p\rangle$ (let $\hbar=M=\omega=1$)

$$|x\rangle=\pi^{-3/4}\exp[-\tfrac{1}{2}x_l^2+\sqrt{2}x_la_l^\dagger-\tfrac{1}{2}a_l^{\dagger 2}]|000\rangle \tag{25}$$

$$|p\rangle=\pi^{-3/4}\exp[-\tfrac{1}{2}p_l^2+\sqrt{2}ip_la_l^\dagger+\tfrac{1}{2}a_l^{\dagger 2}]|000\rangle \tag{26}$$

we can go a crucial step further to recast (23) and (24) into

$$\pi^{-3/2}\int \mathrm{d}x:\exp[-(x_l-\hat{x}_l)^2]:=1. \tag{27}$$

$$\pi^{-3/2}\int \mathrm{d}p:\exp[(-p_l-\hat{P}_l)^2]:=1. \tag{28}$$

Then using the following integral formula:

$$\int \mathrm{d}x\exp[-x_lN_{lm}x_m+V_lx_l]=\pi^{3/2}(\det N)^{-1/2}\exp[\tfrac{1}{4}V_lN_{lm}^{-1}V_m] \qquad N=\tilde{N} \tag{29}$$

where N stands for a positive definite 3×3 matrix, and the IOWP, we can expand

$$\begin{aligned}\exp[\hat{x}_l\Lambda_{lm}\hat{x}_m]&=\int \mathrm{d}x\,\mathrm{e}^{\hat{x}_l\Lambda_{lm}\hat{x}_m}|x\rangle\langle x|\\&=\int \mathrm{d}x\,\pi^{-3/2}:\exp[-x_l^2+\sqrt{2}x_l(a_l^\dagger+a_l)+x_l\Lambda_{lm}x_m-\hat{x}_l^2]:\\&=\int \mathrm{d}x\,\pi^{-3/2}:\exp[-x_l(\mathbb{1}-\Lambda)_{lm}x_m+2x_l\hat{x}_l-\hat{x}_l^2]:\\&=[\det(1-\Lambda)]^{-1/2}:\exp[\hat{x}_l(\mathbb{1}-\Lambda)_{lm}^{-1}\hat{x}_m-\hat{x}_l^2]:\end{aligned} \tag{30}$$

where the matrix $\mathbb{1}-\Lambda$ should be positive definite. Similarly, we have

$$\begin{aligned}\exp[\hat{P}_l\Omega_{lm}\hat{P}_m]&=\int \mathrm{d}p\,\mathrm{e}^{\hat{P}_l\Omega_{lm}\hat{P}_m}|p\rangle\langle p|\\&=\int \mathrm{d}p\,\pi^{-3/2}:\exp[-p_l(1-\Omega)_{lm}p_m+2p_l\hat{P}_l-\hat{P}_l^2]:\\&=[\det(\mathbb{1}-\Omega)]^{-1/2}:\exp[\hat{P}_l(\mathbb{1}-\Omega)_{lm}^{-1}\hat{P}_m-\hat{P}_l^2]:.\end{aligned} \tag{31}$$

4. Applications

The above-derived normally ordered expansion of (13) can greatly simplify the calculation of determining the normalisation factor of some state vectors. For example, let us first consider the following state:

$$|\lambda, \rho\rangle \equiv c \exp[\lambda a_1^\dagger a_2^\dagger + \rho a_2^\dagger a_3^\dagger]|000\rangle \qquad |\lambda|^2 + |\rho|^2 < 1 \tag{32}$$

where c is the normalisation factor to be determined. Using (18) and the property of normal product we immediately have

$$1 = \langle\lambda, \rho|\lambda, \rho\rangle = |c|^2 \langle 000| \exp[\lambda^* a_1 a_2 + \rho^* a_2 a_3] \exp[\lambda a_1^\dagger a_2^\dagger + \rho a_2^\dagger a_3^\dagger]|000\rangle = \frac{|c|^2}{1 - |\lambda|^2 - |\rho|^2} \tag{33}$$

which leads to

$$|c| = (1 - |\lambda|^2 - |\rho|^2)^{1/2}.$$

When $\rho \equiv 0$, $\lambda \equiv \tanh r$, $c = \operatorname{sech} r$, (32) reduces to the well known two-mode squeezed vacuum state [11]. We now give another example by evaluating the normalisation factor of the following state vector with a real parameter γ:

$$\begin{aligned} |\psi\rangle &= c \exp[\gamma(a_1^{\dagger 2} + a_2^{\dagger 2} + a_3^{\dagger 3} - a_1^\dagger a_2^\dagger - a_2^\dagger a_3^\dagger - a_1^\dagger a_3^\dagger)]|000\rangle \\ &= c \exp\left[(a_1^\dagger \quad a_2^\dagger \quad a_3^\dagger) w \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \end{pmatrix} \right] |000\rangle \end{aligned} \tag{34}$$

$$w = \gamma \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

This evaluation would be difficult if we did not have the formula (13). Fortunately, we can now use (13) to directly obtain

$$1 = \langle\psi|\psi\rangle = |c|^2 [\det(\mathbb{1} - 4w^2)]^{-1/2} = |c|^2/(1 - 9\gamma^2). \tag{35}$$

Therefore, the normalisation factor is

$$|c| = (1 - 9\gamma^2)^{1/2}.$$

Operating with a_1, a_2, a_3 on $|\psi\rangle$, respectively, we have

$$\begin{aligned} a_1|\psi\rangle &= \gamma(2a_1^\dagger - a_2^\dagger - a_3^\dagger)|\psi\rangle \\ a_2|\psi\rangle &= \gamma(2a_2^\dagger - a_1^\dagger - a_3^\dagger)|\psi\rangle \\ a_3|\psi\rangle &= \gamma(2a_3^\dagger - a_1^\dagger - a_2^\dagger)|\psi\rangle. \end{aligned} \tag{36}$$

There now follow the three independent equations

$$\frac{1}{\sqrt{3}}(a_1 + a_2 + a_3)|\psi\rangle = 0 \tag{37}$$

$$[(a_1 - a_2) - 3\gamma(a_1^\dagger - a_2^\dagger)]|\psi\rangle = 0 \tag{38}$$

$$\left[\left(a_3 - \frac{a_1 + a_2}{2}\right) - 3\gamma\left(a_3^\dagger - \frac{a_1^\dagger + a_2^\dagger}{2}\right)\right]|\psi\rangle = 0. \tag{39}$$

By setting

$$\gamma \equiv \frac{\omega - \bar{\omega}}{3(\omega + \bar{\omega})}$$

we get

$$1 - 9\gamma^2 = \frac{4\omega\bar{\omega}}{(\omega + \bar{\omega})^2}$$

so the state $|\psi\rangle$ becomes

$$|\psi\rangle = \frac{2\sqrt{\bar{\omega}\omega}}{\omega + \bar{\omega}} \exp\left(\frac{\omega - \bar{\omega}}{3(\omega + \bar{\omega})} (a_1^{\dagger 2} + a_2^{\dagger 2} + a_3^{\dagger 2} - a_1^\dagger a_2^\dagger - a_2^\dagger a_3^\dagger - a_1^\dagger a_3^\dagger)\right)|000\rangle \tag{40}$$

which is just the ground state [12] of the three-coupled oscillator whose Hamiltonian is given by [13] (here we recover M and ω)

$$H = \frac{\hat{P}_i^2}{2m} + \frac{M}{2}\omega^2 \hat{x}_i^2 + \frac{K}{2}[(\hat{x}_1 - \hat{x}_2)^2 + (\hat{x}_2 - \hat{x}_3)^2 + (\hat{x}_3 - \hat{x}_2)^2] \tag{41}$$

and ω is related to $\bar{\omega}$ by $\bar{\omega}^2 = \omega^2 + 3K/M$.

In summary, we see that the IWOP technique can greatly simplify the operation of normally ordering some multimode exponential operators. Needless to say, the formalism in this work can also be generalised to discuss the case of fermion operators, since we have already introduced the IWOP technique into the fermionic system [14].

References

[1] Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)
Wilcox R M 1967 *J. Math. Phys.* **8** 962
[2] Fan Hong-yi and Ruan Tu-nan 1984 *Sci. Sin.* A **27** 391
[3] Fan Hong-yi, Zaidi H R and Klauder J R 1987 *Phys. Rev.* D **35** 1831
[4] Fan Hong-yi and Ren Yong 1988 *J. Phys. A: Math. Gen.* **21** 1971
Fan Hong-yi and Klauder J R 1988 *J. Phys. A: Math. Gen.* **21** L725
Fan Hong-yi and Vander Linde J 1989 *Phys. Rev.* A **39** 2987
[5] Klauder J R and Bo-Sture Skargerstam 1985 *Coherent States* (Singapore: World Scientific)
[6] Fan Hong-yi and Ruan Tu-nan 1985 *Commun. Theor. Phys. (Beijing)* **4** 181
[7] Berezin F A 1966 *The Method of Second Quantisation* (New York: Academic)
[8] Joshi A W 1975 *Matrices and Tensors in Physics* (New Delhi: Wiley Eastern) p 63
[9] Fan Hong-yi 1989 *J. Phys. A: Math. Gen.* **22** 1193
[10] Dirac P A M 1958 *The Principle of Quantum Mechanics* (Oxford: Clarendon)
[11] Schumaker B L 1985 *Phys. Rep.* **135** 317
[12] Fan Hong-yi 1989 *Int. J. Quantum Chem.* **35** 585
[13] Saxon D S 1968 *Elementary Quantum Mechanics* (New York: Holden-Day)
[14] Fan Hong-yi 1989 *J. Phys. A: Math. Gen.* **22** 3423